Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solutions for Problem Set 8

Problem 1 (3 points)

(a) The filter $H(z)$ is a first order all-pole filter. The output of the filter is therefore an $AR[1]$ process.

(b) The first order predictor is defined as

$$\hat{x}(n) = -a_1 x(n-1)$$

The prediction coefficient $-a_1$ can be found by minimizing the prediction error power $\sigma_f^2 = E[f^2(n)]$, where $f(n) = x(n) - \hat{x}(n)$ is the prediction error. We have that

$$\sigma_f^2 = E[(x(n) - \hat{x}(n))^2] = E[(x(n) + a_1 x(n-1))^2]$$

The optimal value for a_1 can be found by minimizing the prediction error power, i.e.

$$\frac{\partial \sigma_f^2}{\partial a_1} = 0.$$

Thus, we get

$$E\left[2(x(n) + a_1 x(n-1))x(n-1)\right] = 0$$
$$\gamma_{xx}(1) + a_1 \gamma_{xx}(0) = 0 \Rightarrow a_1 = -\frac{\gamma_{xx}(1)}{\gamma_{xx}(0)}$$

In the previous problem set the autocorrelation function of the signal $x(n)$ was found to be

$$\gamma_{xx}(m) = \left(-\frac{1}{2}\right)^{|m|}.$$

Thus, we get

$$a_1 = \frac{1}{2}.$$

By repeating the procedure for the second order predictor, we get

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_2 = 0.$$

This means thet we can not obtain further reduction of the prediction error by using a higher order peredictor.

The above results could be expected, since $x(n)$ is an AR[1] process. The optimal predictor is thus the first order predictor with prediction coefficient equal to the filter coefficient.

Problem 2

(a) This is a MA(1)-process, as only the current and the former value of the input signal are used in forming the output signal.

(b) We have

$$\begin{aligned} x(n)x(n-l) &= \Big(w(n) - 0.4w(n-1)\Big)\Big(w(n-l) - 0.4w(n-l-1)\Big) \\ &= w(n)w(n-l) - 0.4w(n)w(n-l-1) - 0.4w(n-1)w(n-l) \\ &\quad + 0.16w(n-1)w(n-l-1). \end{aligned}$$

Taking the expectation of this leads to

$$\begin{aligned} \gamma_{xx}(l) &= \gamma_{ww}(l) - 0.4\gamma_{ww}(l+1) - 0.4\gamma_{ww}(l-1) + 0.16\gamma_{ww}(l) \\ &= 1.16\gamma_{ww}(l) - 0.4\Big(\gamma_{ww}(l+1) + \gamma_{ww}(l-1)\Big) \\ &= 1.16\sigma_w^2\delta(l) - 0.4\sigma_w^2\Big(\delta(l+1) + \delta(l-1)\Big) \\ &= \begin{cases} 1.16 & l = 0, \\ -0.4 & l = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The expression for the power density spectrum $\Gamma_{xx}(f)$ can be found as follows.

$$\begin{aligned} \Gamma_{xx}(f) &= \sum_{l=-1}^{1} \gamma_{xx}(l)e^{-j\omega l} \\ &= -0.4e^{j\omega} + 1.16 - 0.4e^{-j\omega} \\ &= 1.16 - 0.8\cos(2\pi f) \end{aligned}$$

(c) The optimal predictor of order p is given by:

$$\hat{x}(n) = -\sum_{k=1}^{p} a_k x(n-k)$$

To obtain simple matrix equations that we can solve using Matlab, we use the version of the Yule-Walker that does not contain σ_f^2, i.e.

$$\gamma_{xx}(0)a_1 = -\gamma_{xx}(-1)$$

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\gamma_{xx}(-1) \\ -\gamma_{xx}(-2) \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) & \gamma_{xx}(2) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-2) & \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\gamma_{xx}(-1) \\ -\gamma_{xx}(-2) \\ -\gamma_{xx}(-3) \end{bmatrix}$$

for order one, two, and three respectively. Then we calculate σ_f^2 as

$$\sigma_f^2 = \sum_{k=0}^{p} a_k \gamma_{xx}(k).$$

The following Matlab code can be used to find the coefficients, and σ_f^2 for each AR order.

```
gamma_xx = [1.16 -.4 0 0];

R1 = [1.16];
R2 = [1.16 -.4; -.4 1.16];
R3 = [1.16 -.4 0 ; -.4 1.16 -.4 ; 0 -.4 1.16];

a_1 = R1^(-1)*-gamma_xx(2)'
sigma_f1 = sum([1 a_1'].*gamma_xx(1:2))
a_2 = R2^(-1)*-gamma_xx(2:3)'
sigma_f2 = sum([1 a_2'].*gamma_xx(1:3))
a_3 = R3^(-1)*-gamma_xx(2:4)'
sigma_f3 = sum([1 a_3'].*gamma_xx(1:4))
```

Approximated values are given in the table below.

AR order p	Coefficients	σ_f^2
1	$a_1 = 0.35$	1.022
2	$a_1 = 0.39$, $a_2 = 0.14$	1.003
3	$a_1 = 0.39$, $a_2 = 0.16$, $a_3 = 0.05$	1.000

We can see that the mean square error decreases as we increase the model order. This means that the AR model is a better approximation of the MA[1] process.

(d) The power density spectrum estimate obtained by using an AR[p] model

is given by

$$\begin{aligned}\hat{\Gamma}_{ff}(f) &= \Gamma_{ff}(f)|H(f)|^2 \\ &= \sigma_f^2 \left| \frac{1}{A(f)} \right|^2 \\ &= \frac{\sigma_f^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}\end{aligned}$$

Figure 1 shows the power density spectrum estimates based on AR models of different order compared to the power density spectrum of the MA[1] process. It can be seen that the estimates become closer to the correct value as the model order increases. Thus, the AR[3] model is the best approximation of these three.

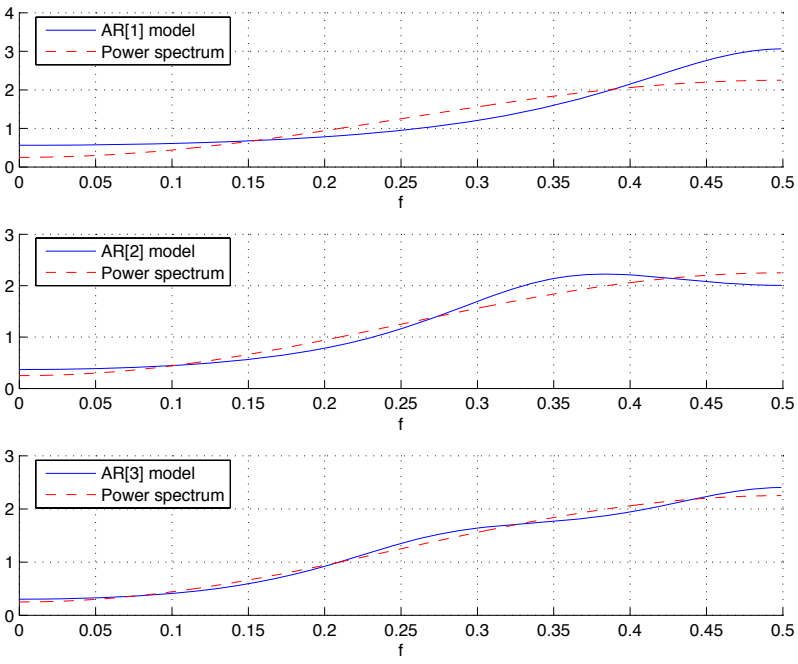

Figure 1: Power density spectrum estimates

Problem 3

- Firstly, the vowel sample files are loaded with *wavread*.
- Then each vowel is modelled as an AR[10] process. The AR coefficients can be found with *lpc*.
- To transform a vowel v_i into another vowel v_j we need the prediction error signal from v_i. For that use $filter$ and the AR coefficients of that vowel. This is the first part of Figure 2:

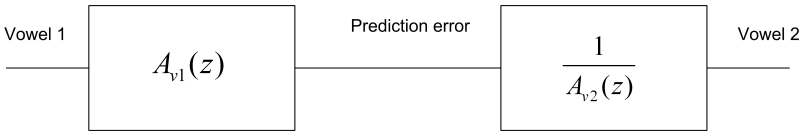

Figure 2: Vowel transformation system

- Generate the new vowel v_j by filtering the prediction error using the inverse prediction-error filter (with the coefficients that you found for v_j). This is the second part of Figure 2.
- The output signal can be played using *sound*.