Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solutions for Problem Set 9

Problem 1

(a) The impulse response can be found as follows.

$$h_d(n) = \text{IDTFT}\{H_d(f)\}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} H_d(f)e^{j2\pi fn}df$$

$$= \int_{-f_c}^{f_c} e^{j2\pi fn}df$$

$$= \frac{1}{2\pi jn}\left[e^{j2\pi fn}\right]_{-f_c}^{f_c}, \quad n \neq 0$$

$$= \frac{e^{j2\pi f_c n} - e^{-j2\pi f_c n}}{2\pi jn}, \quad n \neq 0$$

$$= \frac{\sin(2\pi f_c n)}{\pi n}, \quad n \neq 0$$

For $n = 0$, we get

$$h_d(0) = \int_{-f_c}^{f_c} 1 df = 2f_c.$$

(b) First, in order to obtain a causal filter after windowing, we shift $h_d(n)$ $(N-1)/2$ samples to the right (we assume that N is an odd number). Then, we multiply by $w(n)$, and obtain the following unit sample response

$$h(n) = h_d\left(n - \frac{N-1}{2}\right)w(n) = \begin{cases} \frac{\sin\left[2\pi f_c\left(n - \frac{N-1}{2}\right)\right]}{\pi\left(n - \frac{N-1}{2}\right)} \cdot w(n) & n \neq \frac{N-1}{2} \\ 2f_c \cdot w\left(\frac{N-1}{2}\right) & n = \frac{N-1}{2}. \end{cases}$$

(c) The required MATLAB function can be implemented as follows.

```
function h = oppg1c (w, fc)
N = length(w);
n = (0:N-1)';
h = sin(2*pi*fc*(n-(N-1)/2))./ (pi*(n - (N-1)/2)).*w;
h((N+1)/2) = 2*fc*w((N+1)/2);
% Note that MATLAB indexes from 1 and not 0
end
```

(d) The magnitude response for the filters are shown in Figure 1. By using rectangular window the resulting filter will have smaller transient band but at the same time it will have bigger passband and stopband ripples. The following code is used to test the function in 1 (c) and plot Fig 1.

```
clc
clear all
close all
N = 31; fc = 0.2;
w1=ones(N,1); % Rrectangualr window
w2 = hamming(N); % Hamming windows
h1 = oppg1c(w1, fc);
h2 = oppg1c(w2, fc);

[H1,w1]=freqz(h1,1);
[H2,w2]=freqz(h2,1);
figure()
subplot(2, 1, 1);
plot(w1/2/pi,abs(H1),'k')
ylabel('Magnitude response')
xlabel('f')
title('Rectangular window and design based on 1(a)')
grid on
subplot(2, 1, 2);
plot(w2/2/pi,abs(H2),'k')
ylabel('Magnitude response')
xlabel('f')
title('Hamming window and design based on 1(a)')
grid on
```

(e) We can use the following MATLAB code to generate and plot the magnitude responses of the filters. The results when plotted are equal to those from 1(c).

```
N=31;
fc=0.2;
w1=ones(N,1);
% use rectangular window
h1=fir1(N-1,fc*2, w1);
```
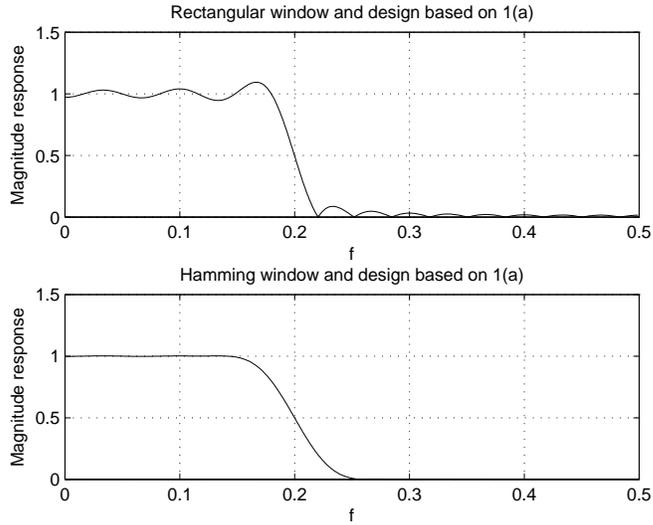
Figure 1: Filters designed using the windowing method from 1(a)

```
% hamming window is default
h2=fir1(N-1,fc*2);
```

Problem 2

(a) The cut-off frequency can be found by solving $|H_a(\Omega_c)| = 1/\sqrt{2}$. Thus, we get

$$\left| \frac{1/RC}{j\Omega_c + 1/RC} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{1/RC}{|j\Omega_c + 1/RC|} = \frac{1}{\sqrt{2}}$$

$$\sqrt{\Omega_c^2 + (1/RC)^2} = \frac{\sqrt{2}}{RC} \Rightarrow$$

$$\Omega_c^2 + \frac{1}{(RC)^2} = \frac{2}{(RC)^2} \Rightarrow$$

$$\Omega_c = \frac{1}{RC},$$

since Ω_c is, by definition, a positive value.

(b) Since $s = j\Omega$ is mapped into $z = e^{j\omega}$ by the bilinear transform, we have that

$$j\Omega = \frac{2}{T} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}.$$

Now, we multiply both the numerator and denominator on the right by
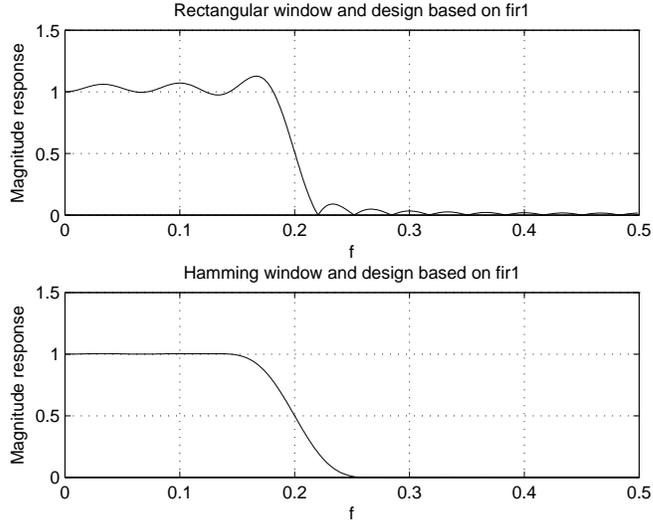
Figure 2: Filters designed using the windowing method using MATLAB's fir1 functions

$e^{j\omega/2}$, and get the following.

$$
\begin{aligned}
\Omega &= \frac{2}{T}\frac{1}{j}\frac{(1-e^{-j\omega})e^{j\omega/2}}{(1+e^{-j\omega})e^{j\omega/2}} \\
&= \frac{2}{T}\frac{1}{j}\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} \\
&= \frac{2}{T}\frac{e^{j\omega/2}-e^{-j\omega/2}}{j}\frac{1}{e^{j\omega/2}+e^{-j\omega/2}} \\
&= \frac{2}{T}\frac{2\sin(\omega/2)}{2\cos(\omega/2)} \\
&= \frac{2}{T}\tan\left(\frac{\omega}{2}\right)
\end{aligned}
$$

(c) We can use the frequency transformation expression to get the following relationship between analog and digital cut-off frequency

$$
\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right).
$$

In addition, we know that $\Omega_c = 1/RC$. Thus, we get the following.

$$\begin{aligned}
H(z) &= H_a(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{1/RC}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + 1/RC} \\
&= \frac{\Omega_c}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \Omega_c} \\
&= \frac{\frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)} \\
&= \frac{\tan\left(\frac{\omega_c}{2}\right)}{\frac{1-z^{-1}}{1+z^{-1}} + \tan\left(\frac{\omega_c}{2}\right)} \\
&= \frac{\tan\left(\frac{\omega_c}{2}\right)(1+z^{-1})}{1 - z^{-1} + \tan\left(\frac{\omega_c}{2}\right) + \tan\left(\frac{\omega_c}{2}\right)z^{-1}} \\
&= \frac{\tan\left(\frac{\omega_c}{2}\right)(1+z^{-1})}{1 + \tan\left(\frac{\omega_c}{2}\right) + (\tan\left(\frac{\omega_c}{2}\right) - 1)z^{-1}} \\
&= \frac{\frac{\tan\left(\frac{\omega_c}{2}\right)}{\tan\left(\frac{\omega_c}{2}\right)+1}(1+z^{-1})}{1 + \frac{\tan\left(\frac{\omega_c}{2}\right)-1}{\tan\left(\frac{\omega_c}{2}\right)+1}z^{-1}}
\end{aligned}$$

By inserting $\omega_c = 0.2\pi$ we get

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.51z^{-1}}.$$

Plots of the magnitude responses of the analog prototype filter, and the resulting digital filter are shown in Figure 3. The magnitude responses are plotted as functions of F and f, respectively.
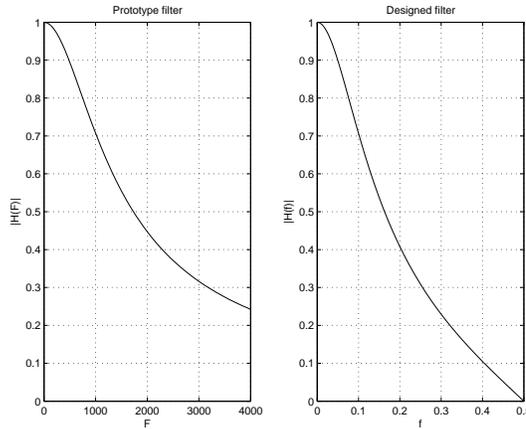
Figure 3: Magnitude responses of the prototype filter and the designed filter

Note that for the analog filter, the frequency axis is not uniquely specified since we have not specified T. For the plot in Figure 3, we have chosen an analog prototype filter with cut-off frequency $F_c = 1000$ Hz ($\Omega_c = 2000\pi$ rad/s). The corresponding value of T can be found as follows.

$$2000\pi = \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right)$$

$$T = \frac{2}{2000\pi}\tan\left(\frac{0.2\pi}{2}\right) \approx 1.75 \cdot 10^{-6}$$

We see that the resulting digital filter is a lowpass filter with a similar shape as the prototype analog filter. However the two magnitude spectra differ especially at higher frequencies due to the nonlinear compression of the physical frequency axis into digital frequencies in the range $[0, \frac{1}{2}]$. Furthermore, the magnitude response of the resulting digital filter is approximately equal to $1/\sqrt{2} \approx 0.707$ at $f_c = 0.1$, which corresponds to $\omega_c = 2\pi f_c = 0.2\pi$, as required in the design specifications. The design specification have thus been met.

Problem 3

(a) The magnitude response of a second order Butterworth filter is given by

$$|H_a(\Omega)| = \frac{1}{\sqrt{1 + (\Omega/\Omega_c)^4}},$$

where Ω_c is the cut-off frequency of the filter. Therefore we have to show that the magnitude response of the given filter can be written as

$$|H_a(\Omega)| = \frac{1}{\sqrt{1 + \Omega^4}},$$

The frequency response is given by

$$H_a(\Omega) = H_a(s)\Big|_{s=j\Omega} = \frac{1}{-\Omega^2 + \sqrt{2}j\Omega + 1},$$

and the magnitude response is thus given by

$$|H_a(\Omega)| = \frac{1}{|-\Omega^2 + \sqrt{2}j\Omega + 1|} = \frac{1}{\sqrt{(1-\Omega^2)^2 + 2\Omega^2}} = \frac{1}{\sqrt{\Omega^4 + 1}}.$$

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Alternatively we can find the cut-off frequency by solving

$$\left| \frac{1}{-\Omega_c^2 + \sqrt{2}j\Omega_c + 1} \right| = \frac{1}{\sqrt{2}}$$

$$\left| -\Omega_c^2 + \sqrt{2}j\Omega_c + 1 \right| = \sqrt{2}$$

$$\sqrt{(1 - \Omega_c^2)^2 + 2\Omega_c^2} = \sqrt{2}$$

$$\sqrt{\Omega_c^4 + 1} = \sqrt{2}$$

$$\Omega_c = 1$$

(b) The plot of the magnitude response is shown in Figure 4. The MATLAB code is given below.

```
num = 1;
den = [1, sqrt(2), 1];

[Ha,Omega]=freqs(num,den);
plot(Omega,20*log10(abs(Ha)))
```
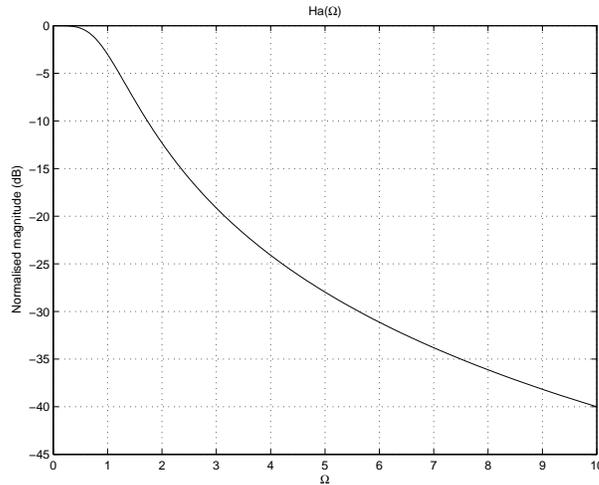

Figure 4: Magnitude response for $H_a(s)$

(c) The poles of the filter are found by solving the following equation

$$s^2 + \sqrt{2}s + 1 = 0.$$

This gives

$$p_1 = \frac{1}{\sqrt{2}}(-1 + j)$$

$$p_2 = \frac{1}{\sqrt{2}}(-1 - j)$$

Note that $p_2 = p_1^*$.

(d) We will now use the impulse invariance method to convert the analog filter to a digital IIR filter.

We can write $H_a(s)$ as

$$H_a(s) = \frac{1}{(s - p_1)(s - p_2)}.$$

By using the partial fraction expansion this can be written as

$$H_a(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2},$$

where

$$c_1 = H_a(s)(s - p_1)\Big|_{s=p_1} = \frac{1}{p_1 - p_2} = \frac{\sqrt{2}}{2j} = -\frac{\sqrt{2}}{2}j.$$

Since $p_2 = p_1^*$, we have that

$$c_2 = c_1^* = -c_1 = \frac{\sqrt{2}}{2}j$$

We can then find $H(z)$ as

$$H(z) = \sum_{k=1}^{2} \frac{c_k}{1 - e^{p_k T}z^{-1}} = \frac{c_1}{1 - e^{p_1 T}z^{-1}} - \frac{c_1}{1 - e^{p_1^* T}z^{-1}}$$

$$= \frac{c_1(e^{p_1 T} - e^{p_1^* T})z^{-1}}{1 - (e^{p_1 T} + e^{p_1^* T})z^{-1} + e^{(p_1 + p_1^*)T}z^{-2}}$$

$$= \frac{\sqrt{2}e^{-\frac{T}{\sqrt{2}}}\sin(\frac{T}{\sqrt{2}})z^{-1}}{1 - 2e^{-\frac{T}{\sqrt{2}}}\cos(\frac{T}{\sqrt{2}})z^{-1} + e^{-\sqrt{2}T}z^{-2}}$$

The constant T is found from the frequency mapping $\omega = \Omega T$, and the requirement that the cut-off frequency of the analogue filter Ω_c should map to the required cut-off frequencies of the digital filters, ω_{c1} and ω_{c2}. This gives

$$T_1 = \frac{\omega_{c1}}{\Omega_c} = 0.25$$

$$T_2 = \frac{\omega_{c2}}{\Omega_c} = 1.2$$

(e) Magnitude responses of the digital IIR filters are shown in Figures 5 and 6 together with the corresponding part of the of the magnitude response of the prototype filter. (Notice that $\omega = \pi$ in Figure 5 corresponds to $\Omega = \frac{\pi}{T_1} = 15.7$ and $\omega = \pi$ in Figure 6 corresponds to $\Omega = \frac{\pi}{T_2} = 2.6$.) The MATLAB code is given below.

```
T = 0.2;
num = [0, sqrt(2)*exp(-T/sqrt(2))*sin(T/sqrt(2))]; %
den = [1, -2*exp(-T/sqrt(2))*cos(T/sqrt(2)), exp(-sqrt(2)*T)];

[H1,w]=freqz(num,den);
plot(w,20*log10(abs(H1)/max(abs(H1))))
```

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(Same for $T = 1.2$)
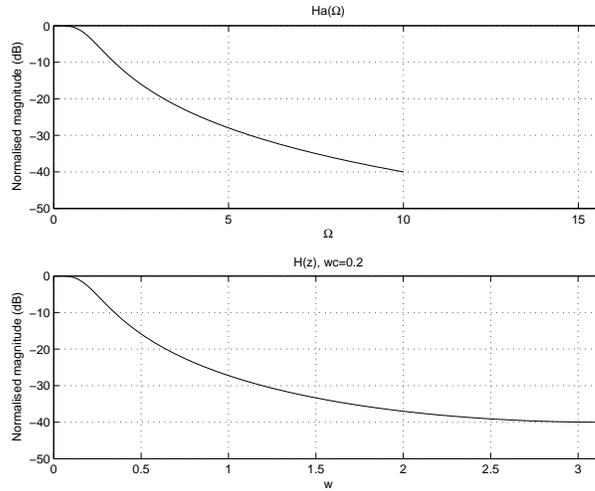
Figure 5: Magnitude response for $H(z)$ with $\omega_c = 0.2$
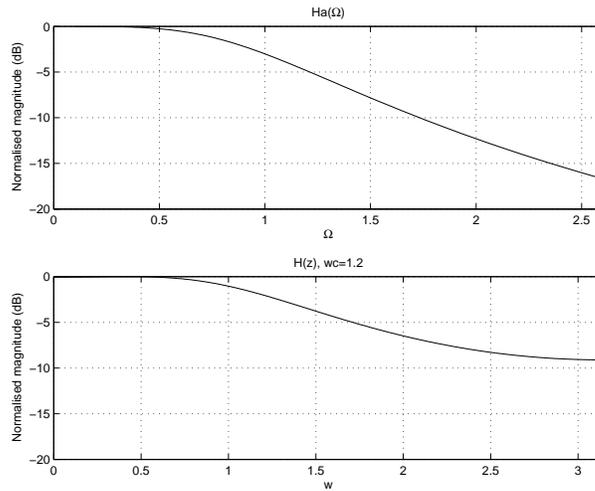
Figure 6: Magnitude response for $H(z)$ with $\omega_c = 1.2$

We can observe that the filter in Figure 5 has a magnitude response that is almost the same as the analog filter. For the other filter with $T = 1.2$ we can see that the attenuation its much lower then the prototype filter. This is due to aliasing in the digital filter. We can also observe that in the filter with $T = 0.2$ the cut-off frequency is correct at $w_c = 0.2$, but for the filter with $T = 1.2$ aliasing has caused the cut-off frequency to shift to approximated $w_c = 1.35$.

Problem 4

(a) Inspecting the magnitude response for the Hamming window method we can see that the specifications are met at an filter order of approximated $N = 360$. Before this we are not able to get a -50dB at 4650Hz. The magnitude response is shown in Figure 7. The right part of the figure shows a closer view of the magnitude response in the passband. The phase response is shown in Figure 8. From this figure we can see that the phase is linear.
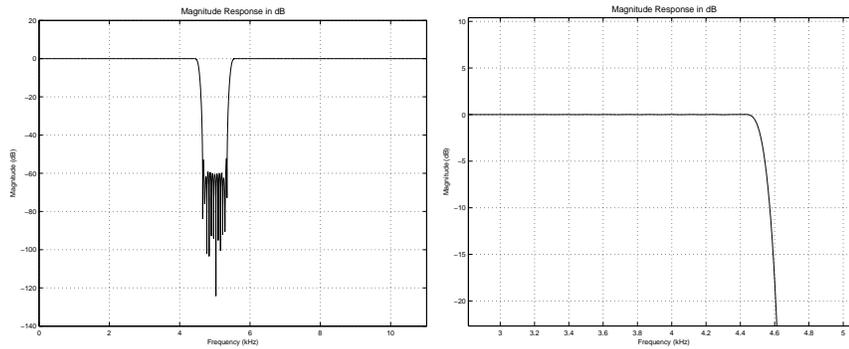
Figure 7: Magnitude response of the FIR filter designed by windowing method (Hamming window). Filter order is 360.
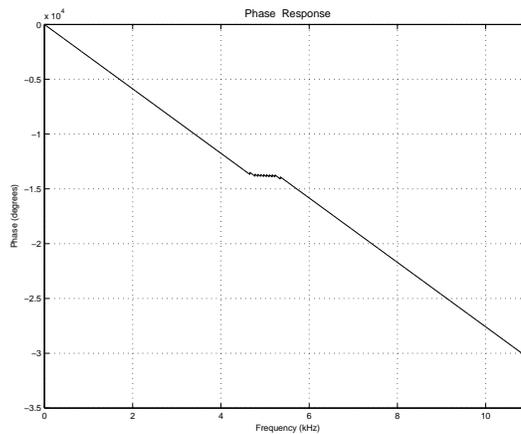
Figure 8: Phase response of the FIR filter designed by windowing method (Hamming window).

(b) We use the `fdatool` and find that the minimum filter order that satisfies the specifications is $N = 200$. The magnitude response of this filter is shown in Figure 9. The right part of the figure shows a closer view of the magnitude response in the passband. The phase response is shown in Figure 10. From this figure we can see that the phase is linear.
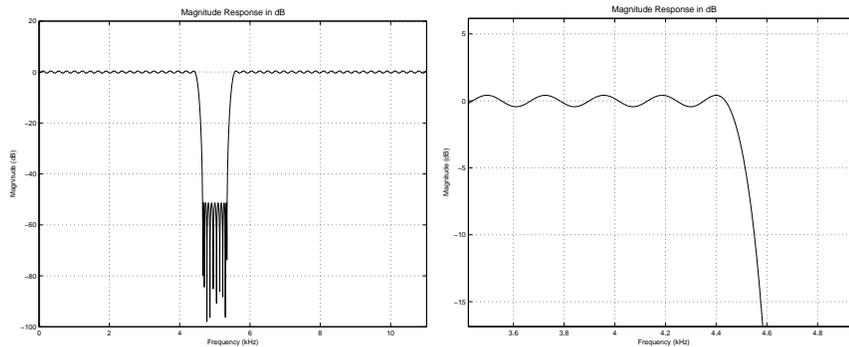
Figure 9: Magnitude response of the FIR filter designed by optimal equripple. Filter order is 200.
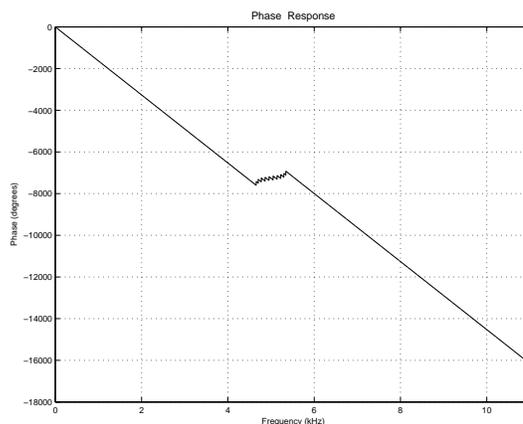
Figure 10: Phase response of the FIR filter designed by optimal equripple.

(c) We use the `fdatool` and find that the minimum filter order that satisfies the specifications is $N = 10$. The magnitude response of this filter is shown in Figure 11. The right part of the figure shows a closer view of the magnitude response in the passband. The phase response is shown in Figure 12.

(d) The windowing method has the advantage that it is very simple. However, it is also the method that results in the filter of highest order. The optimal equiripple design method is more complicated, but it produces a linear phase FIR filter with lowest possible order that meets the given specifications. This is achieved by controlling the size of the errors in the passband and stopband at the same time. Both windowing method and the equiripple method produce filters with linear phase response.

The elliptic IIR filter has significantly lower filter order than both the above methods. Thus, compared to the FIR filters, this filter is less computationally demanding to implement, requires less memory, and
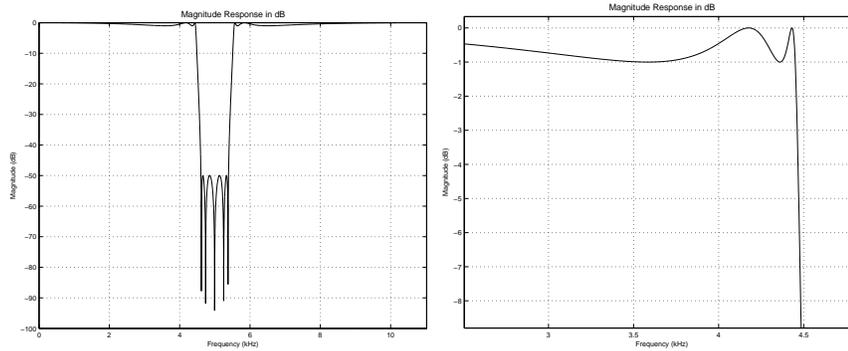
Figure 11: Magnitude response of the elliptic IIR filter. Filter order is 10.
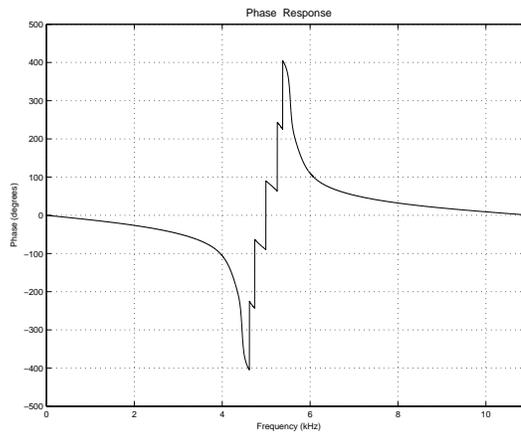
Figure 12: Phase response of the elliptic IIR filter.

allows lower delay. The disadvantage of this filter is that we cannot obtain linear phase response.

(e) We can hear that we remove the noise when we use a high filter length(a high stopband attenuation).

Problem 5

(a) The MATLAB code from Problem 1e) can be used to generate FIR filters of a given length N using a rectangular and a Hamming window.

In the case of rectangular windows, the lowest filter length that satisfies the requirements is $N = 576$. The magnitude response is shown in Figure 13. The right part of the figure shows a closer view of the magnitude response in the passband. As can be seen from these figures, the specifications have been met. The phase response is shown in Figure 14. Because of the high order of the filter, it is difficult to

interpret this plot, but if one looks at a smaller range of frequencies (i.e. zoom in on a part of the plot), one will see that the phase is linear.

In the case of Hamming window, $N = 35$ satisfies the requirements. The magnitude response is shown in Figure 15. The right part of the figure shows a closer view of the magnitude response in the passband. As can be seen from these figures, the specifications have been met. The phase response is shown in Figure 16. From this figure we can see that the phase is linear.
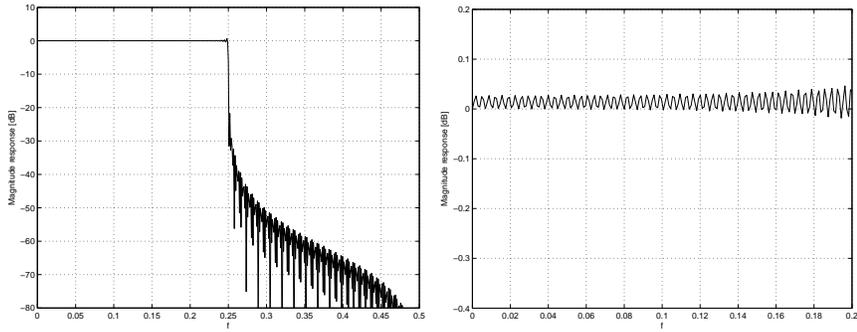
Figure 13: Magnitude response of the FIR filter designed by windowing method (rectangular window). Filter length is 576.
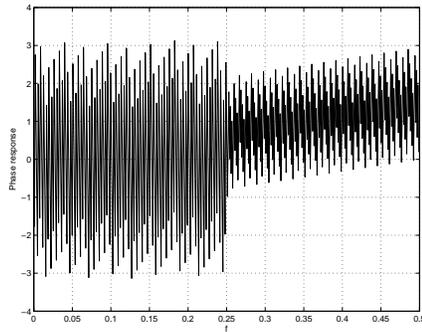
Figure 14: Phase response of the FIR filter designed by windowing method (rectangular window). Filter length is 576.
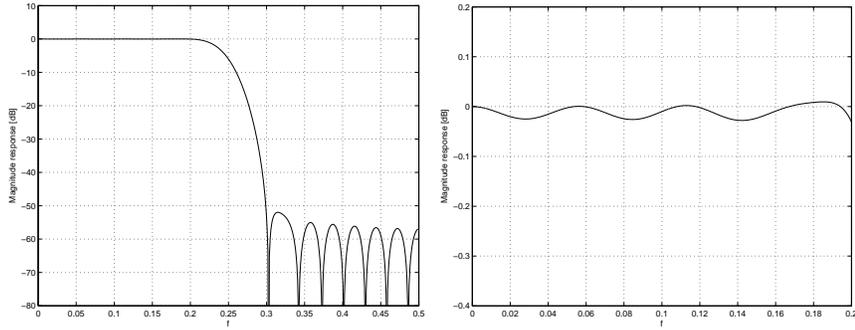
Figure 15: Magnitude response of the FIR filter designed by windowing method (Hamming window). Filter length is 35.
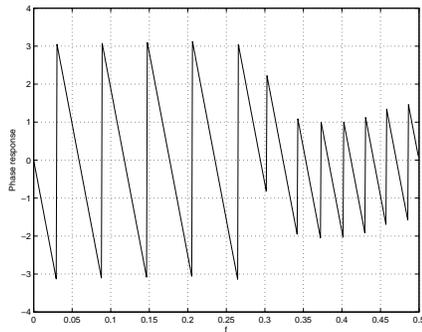
Figure 16: Phase response of the FIR filter designed by windowing method (Hamming window). Filter length is 35.

(b) The optimal linear phase FIR filter is designed using MATLAB function `firpm`. This filter has the lowest possible order among all FIR filters that satisfy the specifications.

The function `firpmord` can give us an estimate of the filter order N. This function can also be used to determine the parameters needed for the `firpm` function. When using this function, we must specify maximum absolute error δ_1 in the passband and stopband. Maximum absolute error in the passband can be calculated from r_p as follows.

$$r_p = 20 \log\left(\frac{1 + \delta_1}{1 - \delta_1}\right)$$

$$\frac{1 + \delta_1}{1 - \delta_1} = 10^{r_p/20}$$

$$\delta_1(1 + 10^{r_p/20}) = 10^{r_p/20} - 1$$

$$\delta_1 = \frac{10^{r_p/20} - 1}{1 + 10^{r_p/20}}$$

$$\delta_1 = \frac{10^{0.4/20} - 1}{1 + 10^{0.4/20}} = 0.023$$

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Maximum absolute error δ_2 in the stopband can be calculated from r_s as follows.

$$r_s = 20 \log \left(\frac{1}{\delta_2} \right)$$

$$\frac{1}{\delta_2} = 10^{r_s/20}$$

$$\delta_2 = \frac{1}{10^{50/20}} = 0.0032$$

The following MATLAB code is used to design the filter.

```
[N,F,a,W] = firpmord([0.2 0.3], [1 0], [0.023 0.0032], 1)
N = 20
h = firpm(N,F,a,W);

[H,w] = freqz(h, 1);
```

The estimated order from `firpmord` is 19, but the resulting filter with this order does not satisfy the specifications (`firpmord` often underestimates the filter order). The minimum filter length that satisfies the specifications is $N = 20$. The magnitude response of this filter is shown in Figure 17. The right part of the figure shows a closer view of the magnitude response in the passband. The phase response is shown in Figure 18. From this figure we can see that the phase is linear.
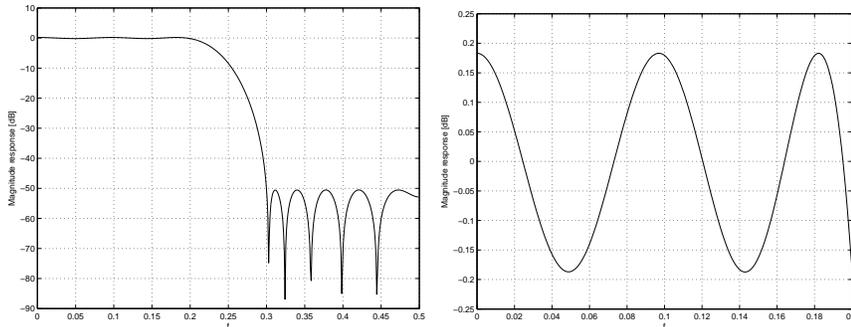
Figure 17: Magnitude response of filter designed with `firpm`
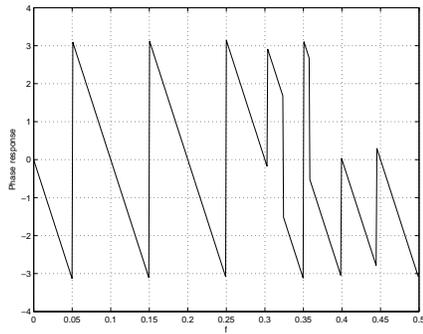
Figure 18: Phase response of filter designed with `firpm`

(c) The following MATLAB code generates an elliptic IIR filter.

```
fp=0.2;
fs=0.3;
rp=0.4;
rs=50;

[N,fc]=ellipord(2*fp,2*fs,rp,rs)
[b,a]=ellip(N,rp,rs,fc);
[H,w]=freqz(b,a);
```

The `ellipord` function estimates that a filter order of $N = 5$ is sufficient, and from the plots of the magnitude response, shown in Figure 19, we can verify that the specifications have been met. The right part of the figure shows a closer view of the magnitude response in the passband.
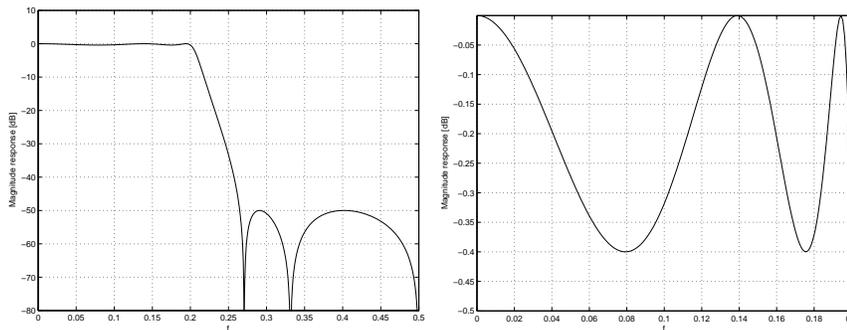
Figure 19: Magnitude response of elliptic IIR filter

(d) The windowing method has the advantage that it is very simple. However, it is also the method that results in the filter of highest order. The optimal equiripple design method is more complicated, but it produces a linear phase FIR filter with lowest possible order that meets the given specifications. This is achieved by controlling the size of the

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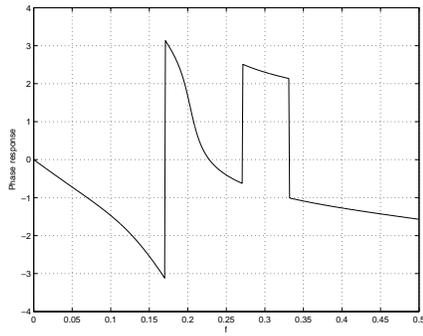
Figure 20: Phase response of elliptic IIR filter

errors in the passband and stopband at the same time. Both windowing method and the equiripple method produce filters with linear phase response.

The elliptic IIR filter has significantly lower filter order than both the above methods. Thus, compared to the FIR filters, this filter is less computationally demanding to implement, requires less memory, and allows lower delay. The disadvantage of this filter is that we cannot obtain linear phase response.